Reprinted from Mathematics Magazine 40.5(1967)284.

Exponential Derivative

653. [March, 1967] *Proposed by by Sam Newman, Atlantic City, New Jersey.*

What is dy/dx of

$$y = \underbrace{x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}}}_{n} \quad ?$$

II. *Solution by Stanley Rabinowitz, Far Rockaway, New York.*

Let f_n denote the function $x^{x^{\cdot^{\cdot^{x}}}}$ (where there are nx's). $(df_n/dx) = f_nf_{n-1}/x + f_n(\ln x)df_{n-1}/dx$. Using this formula, one easily finds that

$$\frac{df_n}{dx} = f_nf_{n-1}/x + f_nf_{n-1}f_{n-2}(\ln x)/x + f_nf_{n-1}(\ln x)^2\frac{df_{n-2}}{dx}\,.$$

Continuing to substitute, one gets by induction

$$\frac{df_n}{dx} = \sum_{j=1}^{k}\left[\frac{(\ln x)^{j-1}}{x}\prod_{i=0}^{j} f_{n-i}\right] + \left[\prod_{i=0}^{k-1} f_{n-i}\right](\ln x)^k\frac{df_{n-k}}{dx}\,.$$

When $k=n-1$, we have

$$\frac{df_n}{dx} = \sum_{j=1}^{n-1}\left[\frac{(\ln x)^{j-1}}{x}\prod_{i=0}^{j} f_{n-i}\right] + (\ln x)^{n-1}\prod_{i=0}^{n-2} f_{n-i}$$